Special Relativity Notes

Emily Nardoni

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This set of notes is a collection of topics relevant to UCSD course 4D, as taught winter quarter 2015. Most of these topics were covered to some extent in recitation sections. Meant to be used as supplementary readings.

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1 Some thought experiments

1.1 Directions perpendicular to a boost

Idea: Use a symmetry argument to show that a meter stick oriented along the x-axis will still be measured to be one meter long in a frame F' which is moving at a constant speed in a direction perpendicular to the x-axis.

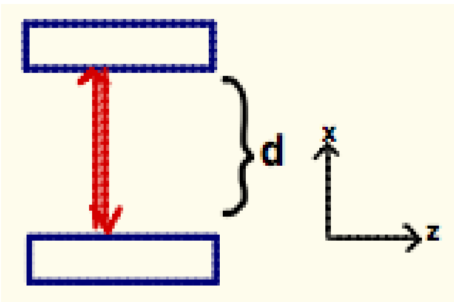
Take two meter sticks. On one attach two blue pens, one at each end, and on the other two red pens. Put the blue one along the x-axis in frame F, and the red one in frame F' moving moving at speed v perpendicular to the x-axis, say along the z-axis. Put everything on a big piece of paper, so that as the meter sticks move the pens draw out the length of the meter stick on the paper.

In frame F from the perspective of the blue meter stick, the red one is moving away at velocity v, and so if the length were to be changed at all it would be contracted, and it would see the red lines being drawn inside its blue lines on the paper. But consider the situation from F', where the red one sees the blue one moving away. If the length were contracted then the blue lines would have to be drawn inside the red ones on the paper. (Clearly it can't be contracted in one frame and expanded in another by equivalence of inertial reference frames.) Thus the only consistent outcome is that the lines are drawn on top of each other, and so the meter stick must be viewed as the same size in both frames.

1.2 Derive time dilation from the invariant interval

Setup: Consider a clock which consists of two mirrors separated by a distance d. Light bounces between them, and the clock ticks whenever light hits one of the mirrors.

Assumptions: The speed of light is constant in all inertial reference frames, lengths aren't changed in directions perpendicular to boost direction.



Call the axis along which the clock is oriented the x-axis. Let's analyze two ticks of the clock from a frame F in which the clock is at rest, and a frame F' which is moving at velocity v in the $(-z)$-direction, perpendicular to the clock. Consider Event 1 to be light leaving one of the mirrors, and Event 2 to be light returning to that mirror after bouncing back from the other mirror.

Light is observed to travel at speed c in both frames, and so the total time elapsed in each frame is the total distance traveled in that frame over c. We know that directions perpendicular to the direction of a boost are unaffected by the boost, so in both frames the total distance the light travels along the x-axis is $2d$. This is the total distance traveled in F, so the total time elapsed in F between Events 1 and 2 is $\Delta t = 2d/c$. In terms of the Events, we choose our coordinate system such that

$$E_1 = (0,0,0,0), \quad E_2 = (2d,0,0,0). \tag{1}$$

However in frame F' the light also travels along the z-direction, so the total distance traveled is given by

$$\text{Distance traveled} \;=\; 2\sqrt{\left(\frac{v\Delta t'}{2}\right)^2 + d^2} = c\Delta t' \tag{2}$$

Solving for $\Delta t'$,

$$\Delta t' = \frac{2d\gamma}{c}, \qquad \gamma \equiv \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}. \tag{3}$$

We chose the time elapsed to be positive, picking out the positive sign when we took the square root. Thus, we have derived time dilation:

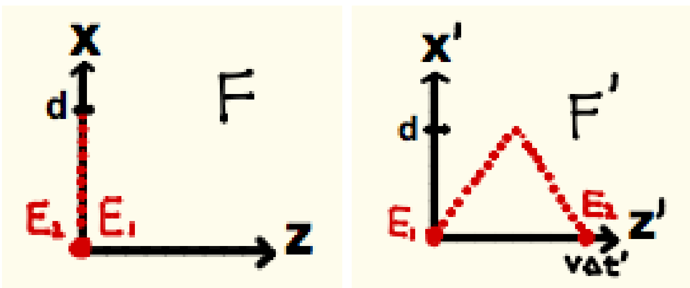
$$\Delta t' = \gamma \Delta t. \tag{4}$$

In terms of the Events, in the Frame F' coordinate system we have

$$E_1' = (0,0,0,0), \quad E_2' = 2d\gamma(1,0,0,\frac{v}{c}). \tag{5}$$

From this discussion, we can immediately see that

$$(c\Delta t)^2 = (c\Delta t')^2 - (\Delta z')^2. \tag{6}$$



We can actually extend this argument to show that the statement holds for all timelike separated events. Consider a new frame F'' with a different boost velocity in the z-direction. By the discussion above, the coordinates of Event 2 in this frame are

$$E_2'' = \frac{2d}{\sqrt{1-\left(\frac{u}{c}\right)^2}}(1,0,0,\frac{u}{c}). \tag{7}$$

Say we wanted to put this event anywhere in the $(\Delta t'', \Delta z'')$ plane with some choice of d and u; in other words, solve

$$\frac{2d}{\sqrt{1-\left(\frac{u}{c}\right)^2}} = c\Delta t'', \quad \frac{2d}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\left(\frac{u}{c}\right) = \Delta z'' \tag{8}$$

$$\Rightarrow d = \frac{1}{2}\sqrt{(c\Delta t')^2 - (\Delta z'')^2}, \quad u = \frac{\Delta z''}{\Delta t''} \tag{9}$$

Recall that $u \in [-c, c]$. Then we see that as long as $(\Delta z'')^2 < (c\Delta t'')^2$—or in other words, the interval is timelike—indeed $u \in [-c, c]$ and d is a physical solution. Thus, by building a sufficiently large light clock and speeding it away arbitrarily fast (up to the speed of light) in the z-direction, the argument we used above will extend to any two timelike separated events, and we will be able to show that in any such frame

$$(\Delta t'')^2 - (\Delta z'')^2 = (\Delta t')^2 - (\Delta z')^2. \tag{10}$$

If we recall that boosts cannot alter the spatial directions perpendicular to the boost, then we arrive at the conclusion that for any two timelike separated events,

$$(\Delta s)^2 = (\Delta s'')^2, \quad (\Delta s)^2 \equiv (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2. \tag{11}$$

1.3 Length contraction

Setup: Observer S is on a train moving at velocity v. An observer S' is on the train platform. There is a device that sends a light pulse from the left hand side of the box-car to the right hand side of the box-car, where is bounces off of a mirror and returns to a photosensor on the LHS. We'd like to time the round trip of the light pulse, and use the measured time to infer the length of the train car.

Assumptions: Constancy of the speed of light in all frames, and time dilation.

On the train, the light pulse travels $2\Delta x = c\Delta t$. On the platform, the time the light travels from emission to the mirror is different from the time traveled back from the mirror to the photodetector because the train is moving during the light's travel time:

$$\Delta t_1' = \frac{(\Delta x' + v\Delta t_1')}{c}, \quad \Delta t_2' = \frac{(\Delta x' - v\Delta t_2')}{c} \tag{12}$$

$$\Rightarrow \Delta t' \equiv \Delta t_1' + \Delta t_2' = \frac{2}{c}\frac{\Delta x'}{1-\frac{v^2}{c^2}} = \frac{2\gamma^2\Delta x'}{c} \tag{13}$$

Using the results from our time dilation thought experiment, we know $\Delta t' = \gamma \Delta t$. Putting this all together, we arrive at the statement of length contraction:

$$\Delta x' = \frac{\Delta x}{\gamma}. \tag{14}$$

1.4 Synchronizing clocks

Idea: One way to define an observer in relativity is by a system of synchronized clocks that store knowledge of events.

Set down a fixed lattice of identical clocks that cover a whole inertial reference frame. In this frame, we would like all these clocks to be synchronized. To do this, call one clock the reference clock. As that clock strikes time 0, it sends out a flash of light as a spherical wavefront. Call the emission of this flash at time 0 the reference event.

When the flash gets to a clock which lies 5 meters away, we want that clock to read 5 light-meters, or $ct = 5$ meters past time 0. Have all the clocks preset to their spatial distance from the reference clock (for instance, a clock which is 50 meters away will prepare its time to be set at $ct = 50$ meters), and as the wavefront passes simultaneously press the button on the clock to set it. In this way, our clocks are synchronized.

Then, when I want to determine the time and location of any event in coordinates of my inertial reference frame, I read off the space position of the event as the location of the clock nearest to it (relative to the reference clock), and the time of the event as the time recorded on that clock when the event happened—imagine stopping the clock nearest an event as an event I want to record occurs. Let the clocks store knowledge of events in this manner.

This system is a good Lorentz observer in relativity. Of course, note that analysis of an event can only occur if information has enough time to get from the clock with the recording of the event to whatever location the analysis is taking place. Even if event information is stored simultaneously in this setup, the information is spread out spatially and must converge at a location for analysis.

1.5 The Twin Paradox

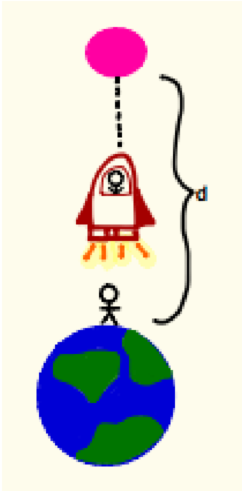
Setup: One twin stays at home on Earth, while the other speeds off at a very large (constant) speed v towards a distant planet, turns around, and heads back to Earth at speed v. The first twin says the trip took time T. How much time elapses according to the traveling twin? [1]

Additionally, say the twins stay in constant touch, and each twin sends radio signals at a frequency f_0 in his own rest frame. How many signals are received by each twin coming from the other?

Assumptions: Assume we can neglect the initial and final accelerations and the turnaround time. We also need the form of the invariant interval, and the relativistic Doppler shift formula.

Let's work in the Earth's coordinate system, (x, t). If the trip took $\Delta t = T$, then the proper time elapsed for the traveling twin will be given by $(cd\tau)^2 = (cdt)^2 - dx^2$, or

[1] We do things differently from how Taylor/Wheeler did them for a supplementary perspective.



$$\Delta\tau = \int dt \sqrt{1 - \left(\frac{dx}{dt}\frac{1}{c}\right)^2} \tag{15}$$

His trajectory follows

$$x(t) = \begin{cases} vt & t < \frac{T}{2} \\ \frac{vT}{2} - vt & t > \frac{T}{2} \end{cases} \tag{16}$$

The velocity $v(t)$ is given by $\frac{dx}{dt} = \pm v$. Plugging into Eq. 15, we find

$$\Delta\tau = T\sqrt{1 - \left(\frac{v}{c}\right)^2} = \frac{T}{\gamma}. \tag{17}$$

Less time passes in the twin's frame, so the twin on Earth has aged more. We still need to answer the paradox: if each twin sees the other twin as moving from his perspective, how come the twin going to the star ages more than the other rather than vice versa?

The answer is that the traveling twin's trajectory involves two different inertial frames, one for the outgoing and one for the returning journeys, which are separated by an acceleration (the turning around). The Earth twin, however, is in the same inertial frame the whole time. Thus there is no paradox.

Now, we need to count how many signals the traveling twin receives during the trip. Using the relativistic Doppler shift formula, if a source emits signals at a frequency f_0 in its rest frame and an observer is moving towards it at velocity v, it observes a frequency f' of

$$f' = f_0\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} \tag{18}$$

On the way to the planet, the traveling twin is moving away from Earth at velocity v, so he sees the signals at a frequency

$$f' = f_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \tag{19}$$

thus receiving a total number of signals:

$$\frac{\Delta\tau}{2}f' = \frac{Tf_0}{2}\left(1-\frac{v}{c}\right). \tag{20}$$

On the way back, he sees the Earth moving toward him at velocity v, resulting in a total number of signals

$$\frac{Tf_0}{2}\left(1+\frac{v}{c}\right). \tag{21}$$

The total number of signals received is f_0T, which is consistent with every signal the first twin emitted during his proper time T reaching the traveling twin.

Counting the number of signals the first twin receives on Earth is the same idea, except that he sees the receding Doppler shift for a time $T/2$ plus the time it takes for light to reach back from the distant planet, $\frac{vT}{2c}$. The traveler sees the ship's signal change from a red-shift to a blue-shift at the midpoint of his journey, while the Earth twin doesn't see this shift until much later.

Thus while the traveling twin is on his way to the planet, the first twin receives

$$\left(\frac{T}{2}+\frac{vT}{2}\right)\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}f_0 \text{ signals} \tag{22}$$

and on the way back the twin receives

$$\left(\frac{T}{2}-\frac{vT}{2}\right)\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}f_0 \text{ signals} \tag{23}$$

for a total of $f_0T\sqrt{1-(\frac{v}{c})^2} = \frac{f_0T}{\gamma}$ signals received. This also makes sense: the traveling twin was broadcasting at frequency f_0 for his whole proper time, which was T/γ, so he ended up sending less signals. All the signals ended up getting to the twin on Earth. It's important that all the counts sent and received are accounted for, since "number of counts" is a time telling mechanism. Biological aging is not different from clock time-keeping.

2 Some derivations/motivations

2.1 Relativistic Doppler shift

Goal: derive the form of the relativistic Doppler shift.

Consider a rocket moving away from me at speed v in my $+\hat{x}$ direction, sending pulses at frequency f' in the rocket's rest frame. Take 2 events to be 2 nodes in the emitted wave, which happen at $E_1' = (ct', x') = (0, 0)$ and $E_2' = (c\tau, 0)$ in the rocket's frame. Here, we've defined τ to be the period of the wave, or $\tau = \frac{1}{f'}$.

Use the Lorentz transformation to find the coordinates of these events in my unprimed frame,

$$\begin{aligned} ct &= \gamma(\beta x' + ct') \\ x &= \gamma(x' + \beta(ct')) \end{aligned} \tag{24}$$

Then, event 1 happens at $E_1 = (ct, x) = (0, 0)$, since we define the origins of the two coordinate systems to coincide, and event 2 happens at $E_2 = (\gamma c\tau, \gamma\beta c\tau)$.

Additionally, light from event 2 takes time $c\Delta t = \Delta x = \gamma\beta c\tau$ to reach me. Thus we don't actually see event 2 until time

$$c\Delta t_{\text{tot}} = (\gamma c\tau) + (\gamma\beta c\tau) = \gamma c\tau(1 + \beta). \tag{25}$$

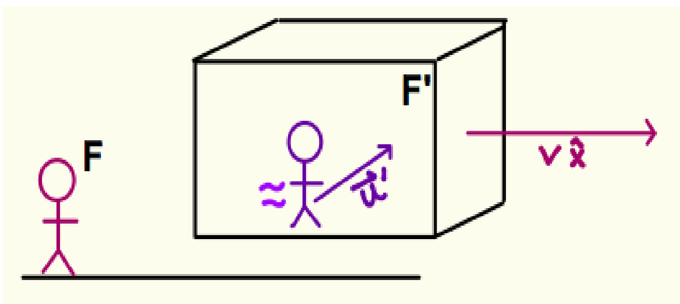
This $c\Delta t_{\text{tot}}$ is precisely the period of the wave as observed in my unprimed frame, thus

$$\tau_{\text{me}} = \frac{1}{f} = \gamma\tau(1 + \beta) = \frac{\gamma}{f'}(1 + \beta). \tag{26}$$

This formula can be algebraically manipulated to the usual form of the relativistic Doppler shift. If the rocket was coming towards me instead of moving away from me, by symmetry we would just take $v \to -v$ and everything else would stay the same.

2.2 Velocity addition formulae

Setup: Consider a frame F and a primed frame F' which is moving at speed v in the $+\hat{x}$ direction relative to F. A person is moving with velocity $\vec{u}'$ in F' (meaning, observers in F' see that person moving at velocity $\vec{u}'$). How does frame F measure the velocity of the person?



What we have is $\vec{u}' = \frac{\Delta x'}{\Delta t'}\hat{x}' + \frac{\Delta y'}{\Delta t'}\hat{y}' + \frac{\Delta z'}{\Delta t'}\hat{z}'$, and what we want is $\vec{u} = \frac{\Delta x}{\Delta t}\hat{x} + \frac{\Delta y}{\Delta t}\hat{y} + \frac{\Delta z}{\Delta t}\hat{z}$. We can use the Lorentz transformation from F' to F,

$$
\begin{aligned}
c\Delta t &= \gamma\left(c\Delta t' + \frac{v\Delta x'}{c}\right)\\
\Delta x &= \gamma\left(\Delta x' + v\Delta t'\right)\\
\Delta y &= \Delta y'\\
\Delta z &= \Delta z'
\end{aligned}
$$

to calculate

$$
\begin{aligned}
u_x &= \frac{\Delta x}{\Delta t} = \frac{\gamma\left(\Delta x' + v\Delta t'\right)}{\gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)} = \frac{\frac{\Delta x'}{\Delta t'} + v}{1 + \frac{v\Delta x'}{c^2\Delta t'}} = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}\\
u_y &= \frac{\Delta y}{\Delta t} = \frac{\Delta y'}{\gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)} = \frac{u'_y}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)}\\
u_z &= \frac{\Delta z}{\Delta t} = \frac{\Delta z'}{\gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)} = \frac{u'_z}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)}.
\end{aligned}
\tag{27}
$$

If the frame F' were moving in the $\hat{-x}$ direction instead, one would just take $v \to -v$ in the formulae above. Here we've left the components of $\vec{u}'$ arbitrary, so if for instance you were given that u'_x were negative (the person is moving in the $-\hat{x}'$ direction in frame F'), you'd attach a minus sign to u'_x in the formulae above.

Since 4-velocity u^μ is a 4-vector, we could also derive these formulae by direct Lorentz transformation on u^μ. To do so, recall that $u^\mu = \gamma_u(c, u_x, u_y, u_z)$, where γ_u takes $|\vec{u}|$ as its argument: $\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$. This is the 4-velocity of the moving person as measured in the unprimed frame F, where u_i are the regular velocities measured in that frame. The Lorentz transformation matrix to take you from frame F' to frame F is

$$
\begin{pmatrix}
\gamma_v & \gamma_v\beta_v & 0 & 0\\
\gamma_v\beta_v & \gamma_v & 0 & 0\\
0 & 0 & 1 & 0\\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \gamma_v \; \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \beta_v = \frac{v}{c}
\tag{28}
$$

for our setup where F' is moving at $+v\hat{x}$ relative to F. Thus we can transform

$$
\begin{pmatrix}
\gamma_u c\\
\gamma_u u_x\\
\gamma_u u_y\\
\gamma_u u_z
\end{pmatrix}
=
\begin{pmatrix}
\gamma_v & \gamma_v\beta_v & 0 & 0\\
\gamma_v\beta_v & \gamma_v & 0 & 0\\
0 & 0 & 1 & 0\\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\gamma_{u'} c\\
\gamma_{u'} u'_x\\
\gamma_{u'} u'_y\\
\gamma_{u'} u'_z
\end{pmatrix}
\tag{29}
$$

From this system of equations, we can solve for

$$\begin{aligned}
\frac{\gamma_u u_x}{\gamma_u c} &= \frac{\gamma_{u'}\gamma_v(u'_x+v)}{\gamma_{u'}\gamma_v(c+\beta_v u'_x)} \;\Rightarrow\; u_x = \frac{u'_x+v}{1+\frac{vu'_x}{c^2}},\\
\frac{\gamma_u u_y}{\gamma_u c} &= \frac{\gamma_{u'} u'_y}{\gamma_{u'}\gamma_v(c+\beta_v u'_x)} \;\Rightarrow\; u_y = \frac{u'_y}{\gamma_v\left(1+\frac{vu'_x}{c^2}\right)},\\
\frac{\gamma_u u_y}{\gamma_u c} &= \frac{\gamma_{u'} u'_y}{\gamma_{u'}\gamma_v(c+\beta_v u'_x)} \;\Rightarrow\; u_y = \frac{u'_y}{\gamma_v\left(1+\frac{vu'_x}{c^2}\right)}
\end{aligned} \tag{30}$$

which are exactly the same as Eq.'s 27.

2.3 Kinetic energy in special relativity

In special relativity, the total energy of a particle is given by $E = \gamma mc^2$, and the rest energy is given by $E_0 = mc^2$. As usual, we define the kinetic energy as the energy of motion caused by doing work on the particle. Claim: Kinetic energy = total energy − rest energy.

Recall that nonrelativistically, one calculates the change in the kinetic energy of a particle by

$$\Delta KE = \int \frac{d\vec{p}}{dt}\cdot d\vec{x} = \int d\vec{p}\cdot\vec{v} = m\int \vec{v}\cdot d\vec{v} = \frac{1}{2}mv^2 - E_0 \tag{31}$$

where E_0 is an integration constant depending on the initial kinetic energy.

Relativistically, we can use $\vec{p} = \gamma m\vec{v}$ to calculate

$$\Delta KE = \int d\vec{p}\cdot\vec{v} = \int d(\gamma m\vec{v})\cdot\vec{v} = m\gamma\vec{v}\cdot\vec{v} - \int m\gamma\vec{v}\cdot d\vec{v} \tag{32}$$

where we used integration by parts to change $\int \vec{v}\cdot d(\vec{p})$ to $\vec{v}\cdot\vec{p} - \int\vec{p}\cdot d\vec{v}$. Change variables to $v' = 1-\frac{v^2}{c^2}$, which don't forget also changes $\gamma \to \gamma' = \frac{1}{\sqrt{v'}}$. Then we can do the remaining integral, and find

$$\begin{aligned}
\Delta KE &= m\gamma v^2 + \frac{mc^2}{\gamma} - E_0\\
&= \gamma mc^2 - E_0.
\end{aligned} \tag{33}$$

Thus, if we define the total energy as $E = \gamma mc^2$, this is indeed of the form total energy − rest energy.

2.4 Relativistic momentum conservation

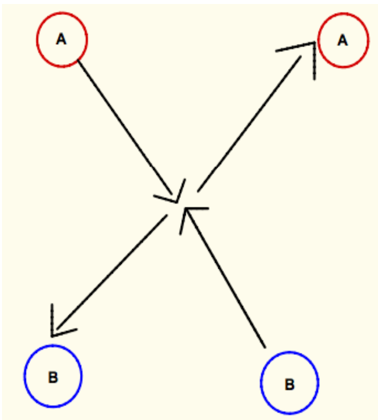
We want a modified version of Newtonian momentum such that we have a useful notion of momentum conservation in relativity. The problem is that while Newtonian momentum may be conserved in one frame, it is not guaranteed to be conserved in another frame.

To motivate the form of relativistic momentum, we consider a general but reasonable modification to $\vec{p} = m\vec{u}$. Since we need to keep the dimensions of momentum, we multiply by some scalar function α, which presumably can (and should) depend on the magnitude of the velocity. Thus we postulate that $\vec{p} = \alpha(|\vec{u}|)m\vec{u}$, for α a scalar function of $|\vec{u}|$.

In particular, consider the limit that $|\vec{u}| \ll c$, in which case in order to be consistent with normal Newtonian momentum conservation we must have $\alpha(|\vec{u}|) \underset{|u|\to 0}{\longrightarrow} 1$. This gives us the condition that $\alpha(0) = 1$.

Now, we consider a relatively simple collision between two balls A and B of equal masses m in two dimensions, which in some frame has the following initial and final velocities:

$$\begin{aligned} \vec{u}^i_A &= u_x\hat{x} - u_y\hat{y}, \qquad \vec{u}^i_B = -u_x\hat{x} + u_y\hat{y} \\ \vec{u}^f_A &= u_x\hat{x} + u_y\hat{y}, \qquad \vec{u}^f_B = -u_x\hat{x} - u_y\hat{y}. \end{aligned} \tag{34}$$



In other words, in this frame the balls come in with the same magnitude but opposite directions of x- and y-velocity, and after the collision continue with the x-component of velocity but switch y-components. Thus classically momentum is conserved in this frame.

We now consider this process occurring in two special frames:

Frame S: where ball A has no x-component of velocity.

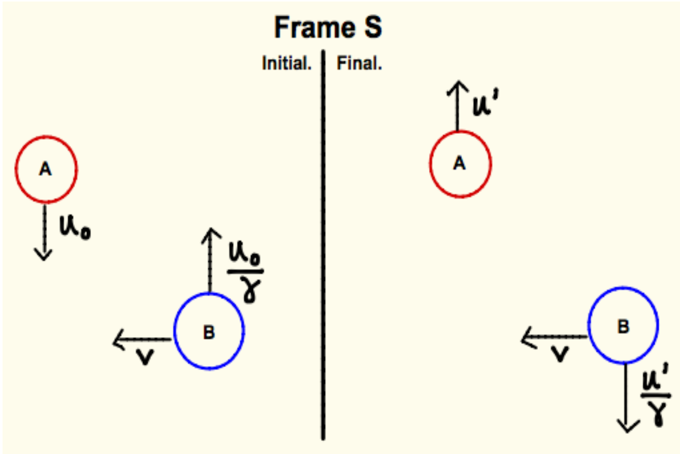
Frame T: where ball B has no x-component of velocity.

Since these frames transform between the x-component rest frame of balls A and B, they are related to each other by a Lorentz transformation in the x-direction. Call the magnitude of the velocity that one frame sees the other moving at in its own x-direction v. Also, call the magnitude of the initial y-component of velocity of the particle in its own x-rest frame u_0.

Then, we can transform the initial y-velocity u_0 in the particle's x-rest frame to the velocity as seen in the other frame which is boosted along the x-direction by the velocity addition formulas. For example, in Frame S the initial y-velocity of B will be given by

$$\text{Ball B:} \quad (u^i_y)^S = \frac{(u^i_y)^T}{\gamma\left(1 + \frac{(u^i_x)^T v}{c^2}\right)} = \frac{u_0}{\gamma} \tag{35}$$

where the second equality holds because by definition $(u_y^i)^T \equiv u_0$, and $(u_x^i)^T = 0$ since in Frame T ball B has no x-component of velocity. The process looks like:



We don't assume that the final y-component velocity u' is u_0. Now, we use conservation of the x-component of momentum in this frame:

$$p_x^i = p_y^i : \quad \alpha(|\vec{u}_i|)mv = \alpha(|\vec{u}_f|)mv \tag{36}$$

$$|\vec{u}_i| = \sqrt{v^2 + \left(\frac{u_0}{\gamma}\right)^2}, \quad |\vec{u}_f| = \sqrt{v^2 + \left(\frac{u'}{\gamma}\right)^2}$$

$$\Rightarrow u_0 = u'. \tag{37}$$

Thus there is no change in the magnitude of the y-component of velocity, and indeed $u' = u_0$. Now use conservation of the y-component of momentum:

$$-\alpha(u_0)mu_0 + \alpha\left(\sqrt{v^2 + \frac{u_0^2}{\gamma^2}}\right)\frac{mu_0}{\gamma} = +\alpha(u_0)mu_0 - \alpha\left(\sqrt{v^2 + \frac{u_0^2}{\gamma^2}}\right)\frac{mu_0}{\gamma}$$
$$\Rightarrow \alpha\left(\sqrt{v^2 + \frac{u_0^2}{\gamma^2}}\right) = \gamma\alpha(u_0). \tag{38}$$

With our consistency condition $\alpha(0) = 1$, this equation becomes $\alpha(v) = \gamma\alpha(0) = \gamma$, or

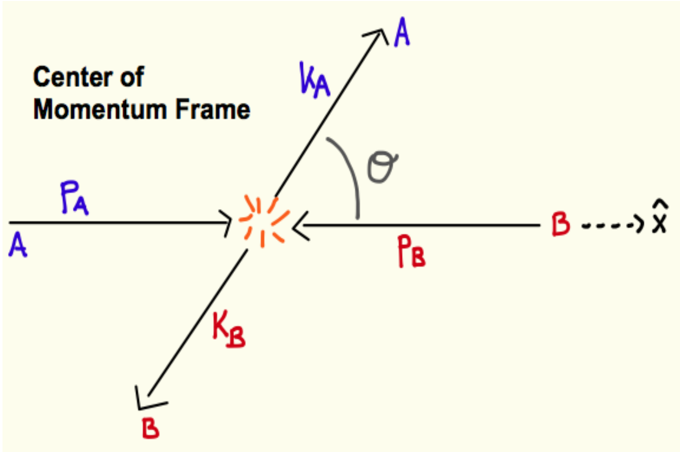
$$\vec{p} = \gamma m\vec{v}, \ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{39}$$

This is in fact the relativistic momentum conserved during interactions in special relativity.

2.5 2-body collision

Let's analyze a relativistic 2-body collision in the center of momentum frame. We won't plug in any numbers, just set up the problem. For simplicity, consider a 2d collision (the particles can move in $\hat{x}$ and

$\hat{y}$). (Thus, when I say 4-vectors below I really mean 3-vectors since we're neglecting the $\hat{z}$-component of spatial momentum.)



Say particle A has mass m_A, and particle B has mass m_B. In the center of momentum frame, the total spatial momentum is zero. Let's choose the x-axis in this frame to coincide with the collision axis, as shown in the figure, and call the magnitude of the initial momentum that each particle has along that axis p. Then the initial momentum 4-vectors are given by:

$$p_A = \left(\frac{E_A}{c}, p, 0\right), \quad p_B = \left(\frac{E_B}{c}, -p, 0\right) \tag{40}$$

and all the relevant formulae are

$$\begin{aligned}
p_A^2 = \left(\frac{E_A}{c}\right)^2 - p^2 = (m_A c)^2, \;\; E_A = \gamma_A m_A c^2, \;\; KE_A = E_A - m_A c^2, \;\; \gamma_A = \frac{1}{\sqrt{1-\frac{v_A^2}{c^2}}} \\
p_B^2 = \left(\frac{E_B}{c}\right)^2 - p^2 = (m_B c)^2, \;\; E_B = \gamma_B m_B c^2, \;\; KE_B = E_B - m_B c^2, \;\; \gamma_B = \frac{1}{\sqrt{1-\frac{v_B^2}{c^2}}} \\
p = \gamma_A m_A v_A = \gamma_B m_B v_B
\end{aligned} \tag{41}$$

where $v_{A,B}$ is the magnitude of the (regular) velocity of the particles in this frame. Given the kinetic energy of a particle we could deduce its total energy, or given its mass and total energy we could deduce its relativistic momentum p, or given its velocity and mass we could deduce p, or so on depending on the problem statement. The total initial energy is $E_A + E_B$, and the total initial spatial momentum is 0. We see that in the special case that $m_A = m_B$, $E_A = E_B$.

After the collision, say particle A goes off at angle θ relative to the collision axis, and by conservation of momentum particle B goes off with opposite momentum. Call the final momentum 4-vectors k^μ, and the final magnitude of momentum p'; then

$$k_A = \left(\frac{E_A^f}{c}, p'\cos\theta, p'\sin\theta\right), \quad k_B = \left(\frac{E_B^f}{c}, -p'\cos\theta, -p'\sin\theta\right) \tag{42}$$

By conservation of energy, $E_A^f + E_B^f = E_A + E_B$. We see that when the two particles have the same mass, $E_A^f = E_B^f = E_A = E_B$. Similarly, the relevant formulae are

$$
\begin{aligned}
k_A^2 = \left(\frac{E_A^f}{c}\right)^2 - p'^2 = (m_A c)^2, \quad E_A^f = \gamma_A^f m_A c^2, \; KE_A^f = E_A^f - m_A c^2, \quad \gamma_A^f = \frac{1}{\sqrt{1 - \frac{(v_A^f)^2}{c^2}}} \\
k_B^2 = \left(\frac{E_B^f}{c}\right)^2 - p'^2 = (m_B c)^2, \quad E_B^f = \gamma_B^f m_B c^2, \; KE_B^f = E_B^f - m_B c^2, \quad \gamma_B^f = \frac{1}{\sqrt{1 - \frac{(v_B^f)^2}{c^2}}} \\
p' = \gamma_A^f m_A v_A^f = \gamma_B^f m_B v_B^f
\end{aligned}
\tag{43}
$$

where v_A^f, v_B^f are the final velocities in this frame.

Now, say we wanted momentum 4-vectors in a frame in which particle B is initially at rest; call it frame R. Since in this frame initially particle B is moving to the left with velocity v_B, this would require a Lorentz transformation in the $+\hat{x}$ direction to a frame moving to the right at v_B. The Lorentz transformations for p_A and p_B, for instance, are

$$
\begin{pmatrix} \frac{E_A^R}{c} \\ p^R \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_B & -\gamma_B \frac{v_B}{c} & 0 \\ -\gamma_B \frac{v_B}{c} & \gamma_B & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{E_A}{c} \\ p \\ 0 \end{pmatrix}, \tag{44}
$$

$$
\begin{pmatrix} \frac{E_B^R}{c} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_B & -\gamma_B \frac{v_B}{c} & 0 \\ -\gamma_B \frac{v_B}{c} & \gamma_B & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{E_B}{c} \\ -p \\ 0 \end{pmatrix} \tag{45}
$$

For instance, say we didn't know the velocity v_B that B would be moving in the center of momentum frame, but we did know E_B and p in that frame. Then we could solve $0 = -\gamma_B(\frac{v_B}{c^2} E_B + p)$ for v_B to do such a transformation.

We see in this frame R, the total initial energy is given by $E_A^R + E_B^R$, which will be conserved in this frame, and the total initial relativistic momentum is $p^R \hat{x}$, which will be conserved.

On the flip side, we could start out with the coordinates of a collision in the frame where B is at rest, frame R, and Lorentz transform to the center of momentum frame with the inverse Lorentz transformation.